Int. J. Solids Structures Vol. 25, No. 9, pp. 1039–1052, 1989
Printed in Great Britain.

0020–7683/89 $3.00 + .00

A THERMOELASTIC THEORY OF FLUID-FILLED POROUS MATERIALS

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(*Received* 9 *August* 1988)

Abstract—By extending the Cleary theory, a thermoelastic theory of fluid-filled porous materials is established, which incorporates the heat transportation by fluid flow through pores in addition to the difference between the thermal expansibility of the pore fluid and that of the solid skeleton. It can be seen that the displacement field is completely coupled with the pore pressure and temperature fields in general; however, for an irrotational displacement, the first field is decoupled from the last two, which are still coupled with each other. Furthermore, after being simplified on the basis of the data on material properties for various types of rock, the field equations are applied to a thermoelastic model problem of hot (or cold) water injection from a spherical cavity into an infinite porous body. The analysis of that problem shows that the effect of the heat transportation by fluid flow on the temperature distribution and the thermal stresses varies greatly according to the kind of rock.

1. INTRODUCTION

Governing equations for a fluid-saturated poroelastic solid in an isothermal quasi-static state have been developed and improved by Biot (1941, 1955, 1956) and Biot and Willis (1957). Inspite of a considerable idealization of the behavior of soils and rocks, this theory has turned out to be rich enough to afford excellent insight into the wide variety of mechanical phenomena. Rice and Cleary (1976) have given a rational reformulation of the Biot theory, and their rationalized version has proved more convenient in solving relevant problems and to interpret the solutions obtained, see Cleary (1976, 1977, 1978) and Rudnicki (1987) for example.

On the other hand, many more elaborate theories have been developed by using the modern theory of mixtures, taking into account finite deformations and thermal effects for most cases, e.g. Crochet and Naghdi (1966) and Bowen (1980). However, it has been pointed out in a discussion by Bazant (1985) that the mixture theory could not demonstrate any advantage over the Biot theory for fluid-saturated porous linear elastic solids and that any new effect could be easily incorporated by a natural extension of the Biot theory or its rationalized version.

Schiffman (1971) has developed an extended Biot theory including the thermal effects. Booker and Savvidou (1984) have derived governing equations which allow for thermal effects. Neither of these theories, however, consider the heat transportation by a fluid flow through pores (i.e. the convective component of heat transfer). In many cases, this heat transportation plays an important and essential role.

The present paper extends the Rice–Cleary (1976) theory so as to incorporate the heat transportation by a pore fluid flow in addition to the effect of difference in expansibility between the pore fluid and the skeletal solid. The extended theory shows that the displacement field is completely coupled with the pore pressure and temperature fields in general; however, for irrotational displacement, the first field is decoupled from the last two, which are still coupled to each other. This pore pressure–temperature coupling involves nonlinearity.

Furthermore, after being simplified on the basis of the data on material properties for various kinds of rock, the field equations are applied to a thermoelastic model problem of hot (or cold) water injection from a spherical cavity into an infinite porous body having a uniform reference temperature. The analysis of this problem reveals that the effect of the heat transportation by fluid flow on the temperature distribution and the thermal stresses varies very much with the kind of rock; that is, that effect is negligible for granites, at least for Charcoal granite, while for sandstone that effect predominates over that of thermal conduction and the high temperature and high thermal stress regions penetrate far away from the cavity.

2. EXTENDED GOVERNING EQUATIONS

By extending the Rice–Cleary (1976) or Cleary (1976, 1977) theory, let us develop a quasi-linear quasi-static thermoelastic theory for a fluid-saturated porous material which is subjected to both mechanical and thermal disturbances. We will incorporate the thermal expansion of the skeletal solid and the pore fluid and the fluid flow induced by the difference in thermal expansibility between them as well as the heat transportation by a fluid flow through the pores. We will start with Cleary's (1976) thesis, where the background for thermodynamic principles for porous media is described in detail; we will give little thermodynamic description here.

2.1. *Constitutive equations*

The material is composed of a solid matrix which contains interstitial pore space filled with a freely diffusing pore fluid. Any trapped fluid is regarded as contributing to the net constitutive behavior of the solid component. Let us denote total stresses, average strains, pore pressure, and change in porosity by σ_{ij}, ε_{ij}, p, Δv, respectively. To incorporate the thermal expansion, we have introduced temperature change $\Delta\theta$ from an initial equilibrium temperature; we use the same temperature for both the matrix solid and pore fluid, because local heat exchange between both components may be rapid enough in comparison with global heat and fluid diffusions.

Cleary has proposed the Gibbs' equation in terms of a new state function [see eqn (16c) in Chapter 2 of Cleary (1976)]. If all inelastic terms are ignored, it gives the basis for deriving constitutive equations for the thermoelastic porous material under consideration; the average strain, pore pressure, and entropy density are expressed by partial derivatives of the state function. We linearize these thermoelastic constitutive laws to obtain

$$\varepsilon_{ij} = C_{ijkl}\sigma_{kl} + B_{ij}p + \alpha_{ij}\Delta\theta \tag{1}$$

$$\Delta v = B_{kl}\sigma_{kl} + Dp + \alpha'\Delta\theta \tag{2}$$

where we have omitted the equations for the entropy density since we will not use it. If the material is isotropic,

$$C_{ijkl} = \frac{1}{4G}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{1+\nu}\delta_{ij}\delta_{kl}\right) \tag{3}$$

$$B_{ij} = \frac{3(\nu_u - \nu)}{2GB(1+\nu)(1+\nu_u)}\delta_{ij} \tag{4}$$

$$D = \frac{1}{B}\left(\frac{1}{K} - \frac{1}{K'_s}\right) - \frac{v_0}{K_f}, \quad K \equiv \frac{2G(1+\nu)}{3(1-2\nu)} \tag{5}$$

$$\alpha_{ij} = \alpha\delta_{ij}. \tag{6}$$

In the above equations, v_0, G, K, ν, and α are the porosity, the shear modulus, the bulk modulus, the Poisson ratio, and the linear thermal expansion coefficient of the porous matrix, respectively. The bulk modulus of the pore fluid is K_f, while K'_s is the effective bulk modulus of the solid constituent, which is regarded as an experimental constant and identical to its bulk modulus under some special conditions as discussed in Rice and Cleary (1976). The induced pore pressure parameter of Skempton and the undrained Poisson ratio are denoted by B and ν_u. When the incompressible model is valid, e.g. for water-saturated soil, $B = 1$ and $\nu_u = 1/2$. In general,

$$0 < B \leqslant 1, \quad 0 < \nu < \nu_u \leqslant 1/2. \tag{7}$$

In eqn (2), α' means the volumetric thermal expansion coefficient of the pore space. If the pores expand with their shapes remaining similar as the porous matrix expands thermally, coefficient α' may be expressed as

$$\alpha' = 3v_0\alpha. \tag{8}$$

Linearization of the pressure–density–temperature curve of the pore fluid yields

$$\Delta\rho/\rho_0 = p/K_f - \bar{\alpha}\Delta\theta \tag{9}$$

where $\bar{\alpha}$ is the volumetric thermal expansion coefficient of the pore fluid and $\Delta\rho$ is a change in fluid density from its reference value ρ_0. A change in fluid mass per unit volume of the material $m = \rho v$ is given by $\Delta m = \rho_0\Delta v + v_0\Delta\rho$. Substituting eqn (9) into this equation, we obtain

$$\Delta m = \rho_0\left(\Delta v + \frac{v_0}{K_f}p - \bar{\alpha}v_0\Delta\theta\right). \tag{10}$$

Next, let us consider constitutive relations associated with the heat and fluid diffusion processes. Denote the heat flux and fluid mass flux by h_i and q_i, respectively, and we can easily obtain the following coupled Fourier-Darcy laws by identifying the consequence of positive entropy production as implied relations between the fluxes and their driving forces [see eqn (14b) in Chapter 2 of Cleary (1976)]:

$$q_i = -\rho_0\kappa_{ij}p_{,j} + L_{ij}\Delta\theta_{,j} \tag{11}$$

$$h_i = -\bar{\kappa}_{ij}\Delta\theta_{,j} + L'_{ij}p_{,j} \tag{12}$$

for the case of no body force. Here, L_{ij} and L'_{ij} are cross-effect coefficients. The former represents the effect called thermo-osmosis. Since the last terms are, in general, much smaller than the first terms on the right-hand sides of eqns (11) and (12), we neglect these terms:

$$q_i = -\rho_0\kappa_{ij}p_{,j} \tag*{(11)$'$}$$

$$h_i = -\bar{\kappa}_{ij}\Delta\theta_{,j}. \tag*{(12)$'$}$$

If the material is isotropic, permeability κ_{ij} and thermal conductivity $\bar{\kappa}_{ij}$ can be reduced to

$$\kappa_{ij} = \kappa\delta_{ij} \tag{13}$$

$$\bar{\kappa}_{ij} = \bar{\kappa}\delta_{ij}. \tag{14}$$

2.2. *Conservation laws*

There are three non-trivial conservation laws in the present quasi-static non-isothermal context: the first is a momentum balance or equilibrium equation

$$\sigma_{ij,j} = 0, \tag{15}$$

if there exists no body force. The second equation is that of fluid mass conservation

$$\frac{\partial \Delta m}{\partial t} + q_{j,j} = 0 \tag{16}$$

for no fluid mass supply. These are the same as those of the isothermal case by Cleary (1977).

The last equation is an energy conservation equation. This can be obtained from eqn (11) in Chapter 2 of Cleary (1976) in the same way as for the conventional linear theory of thermoelasticity, e.g. by Boley and Weiner (1960). Neglecting the terms proportional to $(\varepsilon_{ij})^{\cdot}$ and $(\Delta v)^{\cdot}$ representing the interconvertibility of thermal and mechanical energy, we obtain, for the case of no heat source,

$$\rho^T C^T \frac{\partial \Delta\theta}{\partial t} = -h_{i,i} - (Hq_i)_{,i} \tag{17}$$

where ρ^T and C^T are the total mass density and specific heat, and H means the specific enthalpy of pore fluid. The Lagrangian description is used in eqn (17) and the last term on the right-hand side represents the transportation of enthalpy by fluid flow through pores. For fluids such as water and oil, an increase in internal energy due to a temperature rise may play a predominant role in the enthalpy. Therefore, the specific enthalpy can be expressed as

$$H = C^F \Delta T \tag{18}$$

if we introduce, temporarily, fluid temperature rise ΔT and specific heat C^F of the pore fluid.

On the other hand, we assume that a heat exchange between the matrix solid and the pore fluid is rapid enough compared to the global heat and fluid diffusion processes. Thus, the local heat equilibrium is established:

$$\rho^T C^T \Delta\theta = \rho_0 v_0 C^F \Delta T. \tag{19}$$

By eqns (18) and (19), enthalpy H is expressed in terms of local equilibrium temperature $\Delta\theta$. The last term on the right-hand side of eqn (17), into which the expression of H is substituted, represents the heat transportation by the pore fluid flow. Here, we do not know whether we should keep or omit this nonlinear term; we will discuss this point by reference to the material parameters introduced later for various kinds of rock.

2.3. *Strain–displacement relation*

The deformation of the material is described by strain tensor ε_{ij}, which is defined in terms of a suitable average displacement u_i of the solid constituent as follows:

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{20}$$

3. FIELD EQUATIONS AND IRROTATIONAL DISPLACEMENT FIELD

3.1. *Field equations*

In what follows, attention will be restricted to an isotropic case. Substituting eqns (3), (4), and (6) into (1), and solving it with respect to total stress σ_{ij}, yields

$$\sigma_{ij} = 2G\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right] - \frac{3(\nu_u - \nu)}{B(1+\nu_u)(1-2\nu)}p\delta_{ij} - \frac{2G\alpha(1+\nu)}{1-2\nu}\Delta\theta\,\delta_{ij}. \tag{21}$$

With the help of this equation and eqn (20), equilibrium equation (15) is reduced to a modified Navier equation in terms of displacement as follows.

$$u_{i,jj}+\frac{1}{1-2\nu}u_{j,ji}-\frac{3(\nu_{\mathrm{u}}-\nu)}{GB(1+\nu_{\mathrm{u}})(1-2\nu)}p_{,i}-\frac{2\alpha(1+\nu)}{1-2\nu}\Delta\theta_{,i}=0. \tag{22}$$

By eliminating $\varepsilon_{ii}=u_{i,i}$ from eqn (21) with $i=j$ and eqn (22) differentiated with respect to x_i, we obtain one of the compatability equations in terms of σ_{ii}, p, and $\Delta\theta$. That is

$$\left[\sigma_{ii}+\frac{6(\nu_{\mathrm{u}}-\nu)}{B(1+\nu_{\mathrm{u}})(1-\nu)}p+\frac{4G\alpha(1+\nu)}{1-\nu}\Delta\theta\right]_{,jj}=0. \tag{23}$$

Substituting eqns (4), (5) and (8) into (2) and in turn the resulting equation into eqn (10), we find

$$\Delta m=\frac{3\rho_0(\nu_{\mathrm{u}}-\nu)}{2GB(1+\nu)(1+\nu_{\mathrm{u}})}\left[\sigma_{jj}+\frac{3}{B}p\right]-\rho_0 v_0(\bar{\alpha}-3\alpha)\Delta\theta \tag{24}$$

where use has been made of relations between the material parameters

$$D+\frac{v_0}{k_f}=\frac{1}{B}\left(\frac{1}{K}-\frac{1}{K_s'}\right)=\frac{9(\nu_{\mathrm{u}}-\nu)}{2GB^2(1+\nu)(1+\nu_{\mathrm{u}})}. \tag{25}$$

We can understand the physical meaning of B by taking $\Delta\theta=0$ (isothermal) and $\Delta m=0$ (undrained) in eqn (24) (see Rice and Cleary, 1976). The last term on the right-hand side of (24) represents the effect of the difference in thermal expansibility between the solid and fluid constituents on the fluid mass content. By eliminating $\sigma_{ii,jj}$ from eqn (23) and eqn (24) differentiated twice with respect to x_j, we find

$$p_{,jj}=\frac{2GB^2(1+\nu_{\mathrm{u}})^2(1-\nu)}{9\rho_0(\nu_{\mathrm{u}}-\nu)(1-\nu_{\mathrm{u}})}\left\{\Delta m_{,jj}+\rho_0\left[\frac{6\alpha(\nu_{\mathrm{u}}-\nu)}{B(1+\nu_{\mathrm{u}})(1-\nu)}+v_0(\bar{\alpha}-3\alpha)\right]\Delta\theta_{,jj}\right\}. \tag{26}$$

Use of Darcy's law (11)′ with (13) in mass conservation law (16) gives

$$\frac{\partial\Delta m}{\partial t}=\rho_0\kappa p_{,jj} \tag{27}$$

which is rewritten with the aid of eqn (26) as

$$\frac{\partial\Delta m}{\partial t}=C\left\{\Delta m_{,jj}+\rho_0\left[\frac{6\alpha(\nu_{\mathrm{u}}-\nu)}{B(1+\nu_{\mathrm{u}})(1-\nu)}+v_0(\alpha-3\bar{\alpha})\right]\Delta\theta_{,jj}\right\} \tag{28}$$

with the definition of fluid mass diffusivity

$$C=\frac{2\kappa GB^2(1+\nu_{\mathrm{u}})^2(1-\nu)}{9(1-\nu_{\mathrm{u}})(\nu_{\mathrm{u}}-\nu)}. \tag{29}$$

With the help of eqns (18) and (19) and Fourier's law (12)′ with (14), energy equation (17) is written as

$$\frac{\partial\Delta\theta}{\partial t}=C_0\Delta\theta_{,jj}+C_0'(\Delta\theta p_{,j})_{,j} \tag{30}$$

where C_0 is the thermal diffusivity and C_0' is the parameter concerning the heat transportation by pore fluid flow. These are given by

$$C_0 = \frac{\bar{\kappa}}{\rho^T C^T} \tag{31}$$

$$C'_0 = \frac{\kappa}{c_0}. \tag{32}$$

3.2. *Irrotational displacement field*

Modified Navier equation (22), fluid diffusion equation (28), and thermal diffusion equation (30) derived for a fluid-saturated, isotropic, thermo-poro-elastic body are completely coupled with each other through eqns (21) and (25). However, for the special case of irrotational displacement field, it can be shown that the thermal and fluid diffusion equations, which are still coupled, are decoupled from the deformation field equations.

For the irrotational displacement, it can be expressed as the gradient of a scalar function, that is

$$u_i = \Phi_{,i}. \tag{33}$$

Using this in modifed Navier equation (22) and integrating it with respect to x_i, yields

$$\begin{aligned}\Phi_{,ii} &= \varepsilon_{ii} \\ &= \frac{3(\nu_u-\nu)}{2GB(1-\nu)(1+\nu_u)}p+\frac{\alpha(1+\nu)}{1-\nu}\Delta\theta+g(t)\end{aligned} \tag{34}$$

where $g(t)$ is an arbitrary function of time. Substituting eqn (34) into (21) with $i = j$, we obtain

$$\sigma_{ii} = -\frac{6(\nu_u-\nu)}{B(1-\nu)(1+\nu_u)}p-\frac{4G\alpha(1+\nu)}{1-\nu}\Delta\theta+\frac{2G(1+\nu)}{1-2\nu}g(t). \tag{35}$$

Substituting this into eqn (25) and in turn the resulting equation into eqn (28), we find a fluid diffusion equation in terms of pore pressure,

$$\frac{\partial p}{\partial t} = Cp_{,ii}+C'\frac{\partial\Delta\theta}{\partial t}-C''\frac{\mathrm{d}}{\mathrm{d}t}g(t) \tag{36}$$

where C is given by eqn (29), and C' and C'' by the following:

$$C' = \frac{2GB^2(1+\nu_u)^2(1-\nu)}{9(\nu_u-\nu)(1-\nu_u)}\left[\frac{6\alpha(\nu_u-\nu)}{B(1-\nu)(1+\nu_u)}+c_0(\bar{\alpha}-3\alpha)\right] \tag{37}$$

$$C'' = \frac{2GB(1-\nu)(1+\nu_u)}{3(1-2\nu)(1-\nu_u)}. \tag{38}$$

The second term on the right-hand side of (36) represents the effect of thermal expansion of the solid and fluid constituents on the pore pressure field. Temperature and pore pressure field equations, (30) and (36), are still coupled. Those are decoupled from the deformation field equations, though they are indirectly coupled through traction boundary conditions and function $g(t)$ in some cases.

3.3. *Nondimensional variables and material parameters*

In order to facilitate the analysis in the next section, let us denote the representative length and increases in temperature and pressure by r_0, $\Delta\theta_0$, p_0, respectively, and introduce the following nondimensional variables:

Table 1. Nondimensional material parameters for various rocks

Material	A_0	A'_1	A'_2
Charcoal granite	0.234	12.66	1.220
Westerly granite	0.746×10^{-1}	27.47	1.241
Ruhr sandstone	0.309×10^{-3}	25.47	1.384
Weber sandstone	0.792×10^{-4}	9.89	1.736
Berea sandstone	0.103×10^{-5}	3.79	3.101

$$\Delta\bar{\theta} = \Delta\theta/\Delta\theta_0, \quad \bar{p} = p/p_0, \quad \bar{\sigma}_{ij} = \sigma_{ij}/p_0$$
$$\bar{x}_i = x_i/r_0, \quad \bar{t} = t/(r_0^2/C). \tag{39}$$

By using these variables, eqns (30) and (36) are written as

$$\frac{\partial\Delta\theta}{\partial t} = A_0\frac{\partial^2\Delta\theta}{\partial x_i\,\partial x_i} + A_1\frac{\partial}{\partial x_i}\left(\Delta\theta\frac{\partial p}{\partial x_i}\right) \tag{40}$$

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x_i\,\partial x_i} + A_2\frac{\partial\Delta\theta}{\partial t} - \frac{C''}{p_0}\frac{\mathrm{d}}{\mathrm{d}t}g(t) \tag{41}$$

where bars over all variables have been omitted for simplicity and

$$A_0 = \frac{C_0}{C}, \quad A_1 = \frac{p_0}{G}A'_1, \quad A_2 = \frac{2G(1+\nu)\alpha\Delta\theta_0}{p_0}A'_2 \tag{42}$$

with

$$A'_1 = \frac{1}{\upsilon_0\left[\dfrac{2B^2(1+\nu_u)^2(1-\nu)}{9(1-\nu_u)(\nu_u-\nu)}\right]} \tag{43}$$

$$A'_2 = \frac{1}{2(1+\nu)}\left\{\frac{6(\nu_u-\nu)}{B(1-\nu)(1+\nu_u)} + \upsilon_0\left(\frac{\bar{\alpha}}{\alpha} - 3\right)\right\}\cdot\left[\frac{2B^2(1+\nu_u)^2(1-\nu)}{9(1-\nu_u)(\nu_u-\nu)}\right]. \tag{44}$$

In the above equations, A_0, A'_1, and A'_2, are nondimensional material parameters. Factors p_0/G and $2G(1+\nu)\alpha\Delta\theta_0/p_0$ in the second and last equations of (42) are nondimensional pressure and thermal loads, respectively. For granite and sandstone, values of A_0, A'_1, and A'_2 are listed in Table 1, where the pore fluid is assumed to be water. The values of C, υ_0, ν, ν_u, B are borrowed from Rice and Cleary (1976). For all the rocks, the same thermal diffusivity and thermal expansion ratio are assumed, that is $C_0 = 1.64\times10^{-2}$ cm^2/sec and $\bar{\alpha}/\alpha = 25$. This ratio is roughly evaluated from the data: $\alpha = 8.3\times10^{-6}$ 1/°C for granite, $\alpha = (5 \text{ to } 12)\times10^{-6}$ 1/°C for sandstone, and $\bar{\alpha} = 0.21\times10^{-3}$ 1/°C for water.

4. THERMAL STRESSES CAUSED BY HOT-WATER INJECTION

4.1. *Problem formulation*

One of the simplest applications of the thermo-poro-elasticity theory established in the preceding section is to a spherically symmetric problem of underground perturbations. Consider an infinite fluid-saturated porous elastic solid with a cavity of radius r_0 in equilibrium. Let us find the solution of thermal stresses in this solid caused by a sudden injection of hot (or cold) water into this cavity. The injected water has a temperature higher by $\Delta\theta_0$ relative to the equilibrium temperature and a pressure higher by p_0 compared to the equilibrium pore pressure. It is assumed for simplicity that the cavity wall is completely permeable and that the temperature at the wall is kept the same as that of the injected fluid.

Thus, the initial conditions are given by

$$\sigma_r = 0, \quad \sigma_\theta = 0, \quad \Delta\theta = 0, \quad p = 0 \quad \text{at } t = 0 \tag{45}$$

for all $r \geqslant 1$, and the boundary conditions by

$$\sigma_r = -1, \quad \Delta\theta = 1, \quad p = 1, \quad \text{at } r = 1 \tag{46}$$

$$\sigma_r = 0, \quad \Delta\theta = 0, \quad p = 0 \quad \text{at } r \to \infty \tag{47}$$

for all time $t > 0$, where all bars over the nondimensional variables have been omitted again; this will be the case in what follows.

4.2. *Stresses in terms of pore pressure and temperature*

Because of the spherical symmetry, modified Navier equation (22) can be easily integrated. Using the solution of radial displacement in eqns (20) and (21), and determining the integral constants by the first boundary conditions in (46) and (47), we obtain radial and circumferential stresses

$$\sigma_r = -\frac{1}{r^3}\{1+\pi_0 I_p(r)+\tau_0 I_\theta(r)\} \tag{48}$$

$$\sigma_\theta = \frac{1}{r^3}\{1+\tfrac{1}{2}\pi_0 I_p(r)+\tau_0 I_\theta(r)\} - \tfrac{1}{2}\{\pi_0 p(r)+\tau_0\Delta\theta(r)\} \tag{49}$$

where

$$I_p(r) = \int_1^r \xi^2 p(\xi)\,\mathrm{d}\xi, \quad I_\theta(r) = \int_1^r \xi^2 \Delta\theta(\xi)\,\mathrm{d}\xi \tag{50}$$

and

$$\pi_0 = \frac{6(v_u - v)}{B(1+v_u)(1-v)}, \quad \tau_0 = \frac{2G(1+v)\alpha\Delta\theta_0}{(1-v)p_0}. \tag{51}$$

In eqns (48) and (49), pore pressure p and temperature $\Delta\theta$ depend on time and stresses σ_r and σ_θ do so; but the time dependence has been suppressed. Equations (48) and (49) express stresses in terms of pore pressure and temperature, which will be determined in the next subsection.

4.3. *Pore pressure and temperature*

Pore pressure and temperature are governed by coupled equations (40) and (41), in which $g(t) = 0$ in this specific case because of the infinite extent of the body. These equations are written in the spherical polar coordinate system as

$$\frac{\partial\Delta\theta}{\partial t} = A_0\left(\frac{\partial^2\Delta\theta}{\partial r^2}+\frac{2}{r}\frac{\partial\Delta\theta}{\partial r}\right)+A_1\left\{\frac{\partial\Delta\theta}{\partial r}\frac{\partial p}{\partial r}+\Delta\theta\left(\frac{\partial^2 p}{\partial r^2}+\frac{2}{r}\frac{\partial p}{\partial r}\right)\right\}, \tag{52}$$

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial r^2}+\frac{2}{r}\frac{\partial p}{\partial r}+A_2\frac{\partial\Delta\theta}{\partial t}. \tag{53}$$

As for the nondimensional pressure and thermal loads, those are assumed to be

$$\frac{p_0}{G} = 10^{-3}, \quad \frac{2G(1+v)\Delta\theta_0}{p_0} = 1. \tag{54}$$

This pressure value is quite high and this thermal load corresponds to $\Delta\theta_0 = 35$–80°C for the granite rock and sandstone. For these values, we can simplify coupled equations (52) and (53) for most cases by ignoring (or keeping) some terms in them according to the values of coefficients A_0, A_1, A_2 for each rock.

- For the granitic rocks. We can take $A_1 = 0$ in eqn (52) compared to A_0, at least for Charcoal granite. If we are satisfied with rough estimates of the pore pressure and temperature, and consequently the thermal stresses, this is the case also for Westerly granite. Condition $A_1 = 0$ decouples eqns (52) and (53). Thus, we can easily obtain the analytical solutions for these equations under initial and boundary conditions, (45) and (46), (47). The solutions are

$$\Delta\theta = \frac{1}{r}\,\mathrm{erfc}\left(\frac{r-1}{2\sqrt{A_0 t}}\right) \tag{55}$$

$$p = \left(1+\frac{A_0 A_2}{1-A_0}\right)\frac{1}{r}\,\mathrm{erfc}\left(\frac{r-1}{2\sqrt{t}}\right)-\left(\frac{A_0 A_2}{1-A_0}\right)\frac{1}{r}\,\mathrm{erfc}\left(\frac{r-1}{2\sqrt{A_0 t}}\right) \tag{56}$$

where erfc(·) denotes the complementary error function.

- For the sandstone. We can see that $A_0 = C_0/C$ is very small for all types of sandstone listed in Table 1; this implies that the fluid diffusion is much faster than the thermal one. Thus, we can expect that the fluid diffusion becomes steady before the temperature begins to change, that is fluid diffusion equation (53) is reduced to

$$\frac{\mathrm{d}^2 p}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}p}{\mathrm{d}r}=0, \tag{57}$$

the solution of which is given with the help of boundary conditions (46) and (47) by

$$p = 1/r. \tag{58}$$

By using this solution and new nondimensional time

$$t^* = A_0 t, \tag{59}$$

eqn (52) is written as

$$\frac{\partial \Delta\theta}{\partial t^*} = \frac{\partial^2 \Delta\theta}{\partial r^2}+\frac{2}{r}\frac{\partial \Delta\theta}{\partial r}-2\lambda\frac{1}{r^2}\frac{\partial \Delta\theta}{\partial r} \tag{60}$$

with

$$2\lambda = A_1/A_2. \tag{61}$$

It seems difficult to solve the above differential equation analytically; we cannot avoid resorting to the numerical method. We have employed the Crank–Nicolson implicit method.

- For Berea sandstone. Coefficient λ defined by eqn (61) is extremely large for this particular sandstone so that it is not easy to integrate eqn (60) even numerically. Thus, we will further drop the first two terms on the right-hand side of eqn (60), that is

$$\frac{\partial \Delta\theta}{\partial t^*}+\frac{2\lambda}{r^2}\frac{\partial \Delta\theta}{\partial r}=0. \tag{62}$$

This equation has the exact solution

$$\Delta\theta = H\left(t^* - \frac{1}{6\lambda}(r^3 - 1)\right) \tag{63}$$

where $H(\cdot)$ means the Heaviside unit step function. It can be seen that there is a temperature discontinuity, which proceeds with the speed given by

$$V = \frac{2\lambda}{r^2}. \tag{64}$$

For this particular sandstone, the pore pressure distribution is again given by eqn (58).

5. NUMERICAL CALCULATION AND DISCUSSION

To evaluate the stresses around the cavity caused by the sudden injection of hot (or cold) water into it, it is required to evaluate integrals (50). This evaluation has been done numerically with the exception of the following cases. For all sandstone, the pore pressure is given by eqn (58), so that

$$I_p(r) = \tfrac{1}{2}(r^2 - 1). \tag{65}$$

For the Berea sandstone, the temperature is given by eqn (63). We obtain

$$I_\theta(r) = \begin{cases} \tfrac{1}{3}(r^3 - 1) & \text{for} \quad r \leqslant r_* \\ \tfrac{1}{3}(r_*^3 - 1) & \text{for} \quad r > r_* \end{cases} \tag{66}$$

with

$$r_* = (1 + 6\lambda t^*)^{1/3}. \tag{67}$$

In this particular case, the stresses can be also written in the explicit form: for $r \leqslant r_*$,

$$\sigma_r = -\frac{1}{r^3}\left\{1 + \tfrac{1}{2}\pi_0(r^2 - 1) + \tfrac{1}{3}\tau_0(r^3 - 1)\right\} \tag{68}$$

$$\sigma_\theta = +\frac{1}{r^3}\left\{1 + \tfrac{1}{4}\pi_0(r^2 - 1) + \tfrac{1}{6}\tau_0(r^3 - 1)\right\} - \frac{1}{2}\left(\pi_0 \frac{1}{r} + \tau_0\right) \tag{69}$$

and for $r > r_*$,

$$\sigma_r = -\frac{1}{r^3}\left\{1 + \tfrac{1}{2}\pi_0(r^2 - 1) + \tfrac{1}{3}\tau_0(r_*^3 - 1)\right\} \tag{70}$$

$$\sigma_\theta = +\frac{1}{r^3}\left\{1 + \tfrac{1}{4}\pi_0(r^2 - 1) + \tfrac{1}{6}\tau_0(r_*^3 - 1)\right\} - \frac{1}{2}\left(\pi_0 \frac{1}{r}\right). \tag{71}$$

In the numerical calculation, attention is restricted to the case of "cold water" injection.

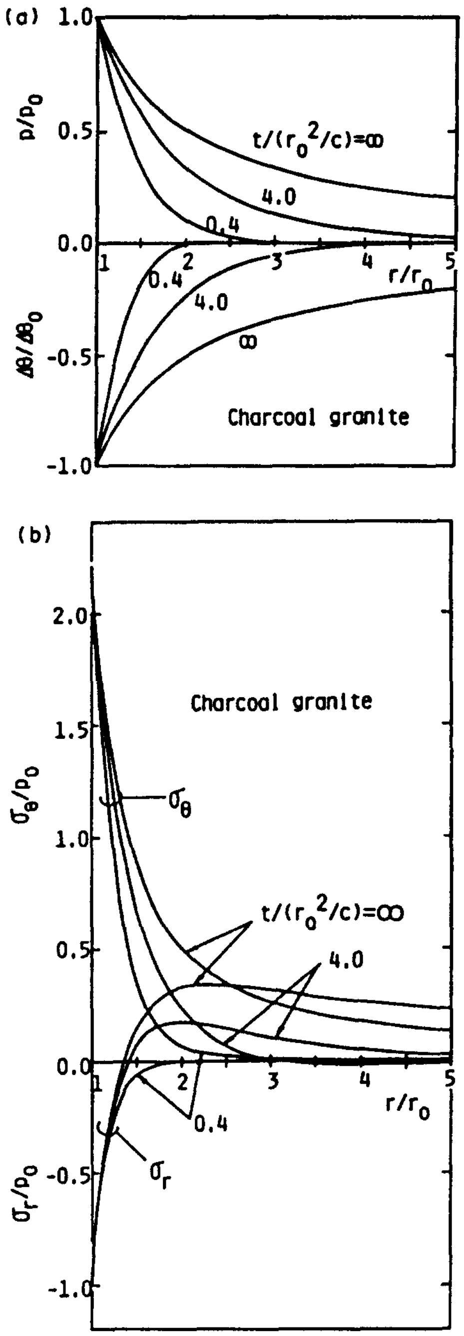

Fig. 1. (a) Pore pressure and temperature and (b) radial and hoop stresses, for Charcoal granite.

The results are plotted in Figs 1, 2, and 3 for Charcoal granite, Ruhr sandstone, and Berea sandstone, respectively. In all the figures, the abscissa is a nondimensional radial distance from the cavity center, with $r/r_0 = 1$ corresponding to the cavity wall. Note that the scales of the nondimensional time in the figures are different from each other for each kind of rock.

Figure 1a shows the pore pressure and temperature distributions at a few different times for Charcoal granite. The pore pressure changes slightly more rapidly than the temperature. Figure 1b is the similar curves for the stresses. Circumferential (or hoop) stress σ_θ at the cavity wall is very high owing to superposition of the stresses due to the thermal and pressure loads. Since the heat transportation by pore fluid flow is negligible

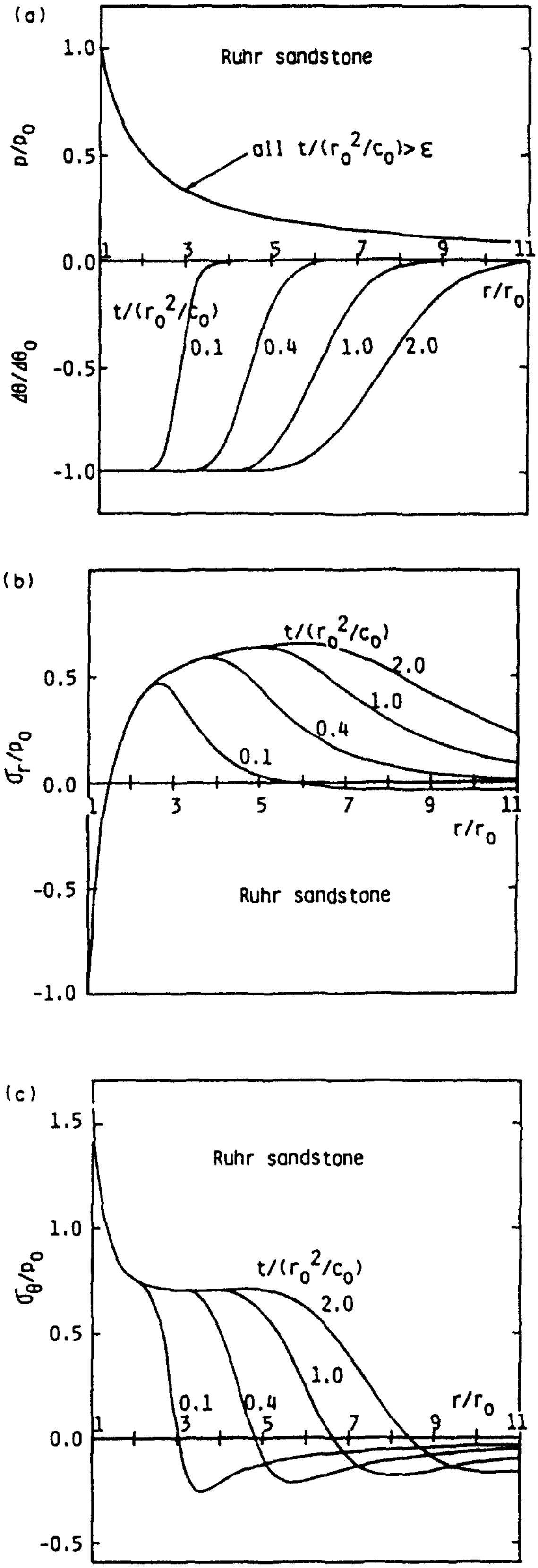

Fig. 2. (a) Pore pressure and temperature, (b) radial stress and (c) hoop stress, for Ruhr sandstone.

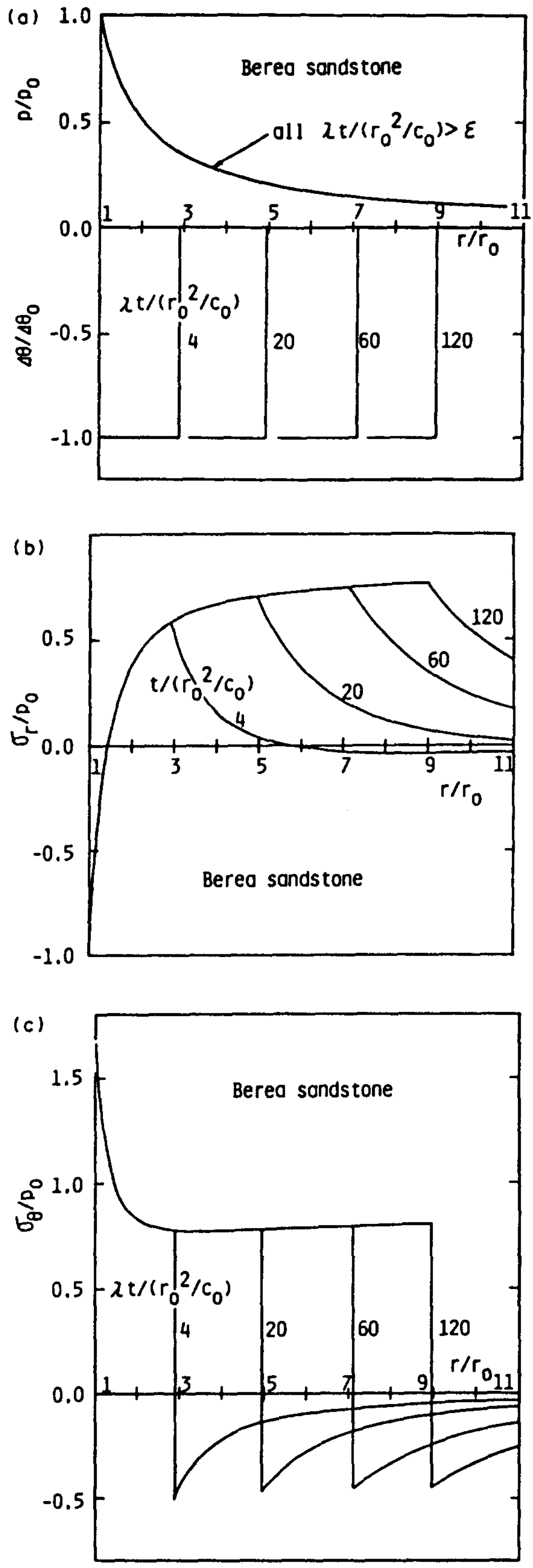

Fig. 3. (a) Pore pressure and temperature, (b) radial stress and (c) hoop stress, for Berea sandstone.

(i.e. $A_1 = 0$), the thermal stresses do not penetrate deep from the cavity even after a long lapse of time.

In contrast to Charcoal granite, Ruhr sandstone has a very large fluid diffusivity compared to its thermal one. The pore pressure, which therefore becomes steady immediately after pressurization at the wall, declines reciprocally proportional to the distance from the cavity. On the other hand, the region of high (but negative for the cold water injection) temperature extends away from the cavity with a time lapse owing to the heat transportation by fluid flow, as shown in Fig. 2a. Corresponding to the extension of the high temperature region, the high tensile thermal stresses are induced in the deep region far from the cavity, as seen in Fig. 2b,c. As the cavity is approached, radial stress σ_r declines up to $-p_0$ and circumferential stress σ_θ rises up to its peak. This is due to the pressure loading at the wall.

Among various types of sandstone, Berea sandstone has extremely large fluid diffusivity relative to its thermal one. Thus, the heat conduction has been ignored. This neglect results in the discontinuous distribution of the temperature and the hoop stress and in the non-smooth distribution of the radial stress, as depicted in Fig. 3. The temperature discontinuity at $r = r_*$ [given by eqn (67)], which may be called a cold water front, proceeds into the infinite body with its jump remaining constant. The jump in hoop stress also remains constant for all time, as follows from eqns (69) and (71).

In conclusion, the high-temperature and high-stress region is restricted to the vicinity of the cavity for a rock having the fluid diffusivity comparable to the thermal one, while such a region penetrates deep into the infinite rock if it has a large fluid diffusivity. In other words, the effect of the heat transportation by a fluid flow on the temperature distribution and the thermal stresses varies very much according to the kind of rock.

It should be added that the same was concluded for a similar model problem of a porous layer cooled and pressurized over its upper surface, though we give no description of it here.

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